# MATH 1A - SOLUTION TO 5.2.67, 5.2.68

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## 1. 5.2.68

We'll do this problem first, because it was on the homework! The next problem is just for practice!

**Problem 1.** *Show that* $f(x) = \frac{1}{x}$ *is not integrable on* $[0, 1]$

Let $f(x) = \frac{1}{x}$, $a = 0$, $b = 1$. Then $x_i = \frac{i}{n}$ and $\Delta(x) = \frac{1}{n}$.

How can we show that something is not integrable? The main point is: **Given** $n$ **we need to CHOOSE a set of points** $x_i^* \in [x_{i-1}, x_i]$ **that 'fails'** (whatever that might mean). As discussed in section, the following choice is a good one:

$$x_1^* = \frac{1}{n^2}$$
$$x_i^* = x_i \qquad \text{for } i \geq 2$$

This works **BECAUSE** $x_1^* \in [x_0, x_1] = [0, \frac{1}{n}]$ and $x_i^* \in [x_{i-1}, x_i]$ (for $i \geq 2$). **Always check this on the exam!**

Because then, we have:

$$\sum_{i=1}^{n} f(x_i^*) \Delta(x) \geq f(x_1^*) \Delta(x) = \frac{1}{\frac{1}{n^2}} \cdot \frac{1}{n} = \frac{n^2}{n} = n$$

Here, we use the fact that every term in the sum is positive, so the sum is greater than its first term $f(x_1^*) \Delta(x)$. Also, $\Delta(x) = \frac{1}{n}$.

And now, if we let $n \to \infty$, the right-hand-side goes to $\infty$, and so by comparison,

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta(x) = \infty$$

So with this choice of $x_i^*$, things have gone awry! The Riemann sum 'blows' up to infinity, and so $f$ is not integrable over $[0, 1]$. The point is: **if a function is integrable, then its integral has to be finite.**

**Other solution:**

Some people wrote up another solution, which is also pretty clever!

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Basically, let $x_i^* = x_i = \frac{i}{n}$ $(1 \leq i \leq n)$, which is in $[x_{i-1}, x_i]$.

Then:

$$\sum_{i=1}^{n} f\left(x_i^*\right) \Delta(x) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n} = \sum_{i=1}^{n} \left(\frac{n}{i}\right) \cdot \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$$

However, some of you might know that:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{i} = \infty$$

It's ok if you don't know this, you're not even supposed to know this because it's covered in Math 1B! (that's why I don't know how many points you would actually get on the exam for this answer...)

And thus:

$$\sum_{i=1}^{n} f\left(x_i^*\right) \Delta(x) = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{i} = \infty$$

And hence $f$ is not integrable on $[0, 1]$ **WARNING:** Note that you **CANNOT** just say

that $f$ is not integrable because it has a vertical asymptote at $x = 0$! For example, the function $g(x) = \frac{1}{\sqrt{x}}$ has a vertical asymptote at $x = 0$, but:

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x}\right]_0^1 = 2$$

Because $2\sqrt{x}$ is an antiderivative of $\frac{1}{\sqrt{x}}$

## 2. 5.2.67

If you need additional practice, here's another problem with solution:

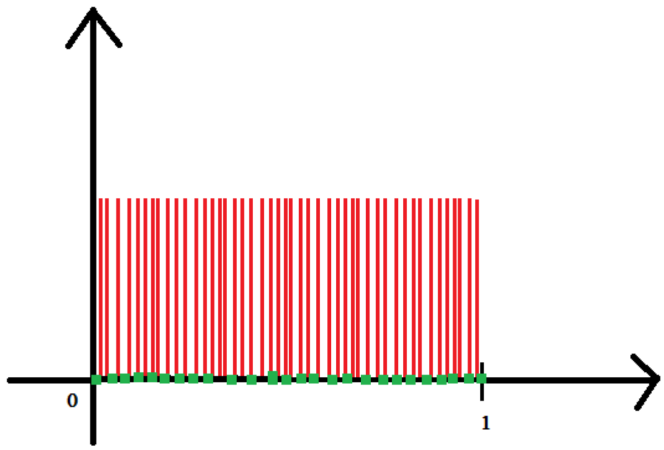
**Problem 2.** *Show that the following function $f$ is not integrable on $[0, 1]$:*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

As usual, let $f(x)$ be as in the problem, $a = 0$, $b = 1$, $x_i = \frac{i}{n}$, and $\Delta(x) = \frac{1}{n}$.

First, let's draw a picture of what's going on:

1A/Nonintegrable.png




In the picture above, the green dots represent where $f(x) = 0$ and the red lines represent where $f(x) = 1$. This problem is unlike the problem above! In this case, the function does not blow up to infinity, but it can't make up its mind! We need to somehow use this fact in order to show that $f$ is not integrable!

But even though this problem is different, the general strategy is almost the same. $f$ integrable means that no matter how we choose the $x_i^*$, we get the same answer! So to show that something is **NOT** integrable, we have to pick two different sets of points $x_i^*$ and $y_i^*$ that give us two different answers!

And here is where we use the fact that $f$ looks the way it does. Namely, let $x_i^*$ be your favorite rational number in $[x_{i-1}, x_i]$ and $y_i^*$ your favorite irrational number in $[x_{i-1}, x_i]$! For example (you don't have to write this, but it's better if you do!), you can choose:

$$x_i^* = x_i = \frac{1}{n}$$
$$y_i^* = \frac{i}{\sqrt{2}n}$$

And you can check that $x_i^* \in [x_{i-1}, x_i]$, and $y_i^* \in [x_{i-1}, x_i]$.

But the point is that $x_i^*$ is rational, and so $f(x_i^*) = 0$ by definition of $f$, and thus the Riemann sum equals to:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta(x) = \lim_{n\to\infty}\sum_{i=1}^{n} 0\cdot\frac{1}{n} = 0$$

And $y_i^*$ is rational, and so $f\left(y_i^*\right) = 1$ by definition of $f$, and thus the Riemann sum equals to:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(y_i^*\right)\Delta(x) = \lim_{n\to\infty}\sum_{i=1}^{n} 1\cdot\frac{1}{n} = \lim_{n\to\infty}\frac{n}{n} = \lim_{n\to\infty} 1 = 1$$

And so we get two different answers (even though we're supposed to get the same answer if $f$ were integrable)!!! Which shows that $f$ is not integrable on $[0,1]$!